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Extension of the Launder, Reece and Rodi model on compressible homogeneous shear flow

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Abstract. This article describes the second order closure progress that was made to calculate compressible homogeneous shear flow with significant compressibility. Several DNS results show that compressibility has an important effect on the pressure-strain correlation. The term recognized as the principal responsible for the change in the magnitude of Reynolds-stress anisotropies. Thus, the pressure-strain incompressible models do not correctly predict compressible turbulence at high-speed shear flow. A method of including compressibility effects in the pressure-strain correlation is the subject of the present study. The concept of the growth rate of turbulent kinetic energy can be used to construct a compressible correction to the Launder, Reece and Rodi model for the pressure-strain correlation. This correction concerns essentially the C_1, C_3 and C_4 coefficients which become in a compressible turbulence situation a function of the turbulent Mach number. The application of the new model shows good agreement with DNS results of Sarkar for cases A_1, A_2 and A_3. These cases correspond to a moderate mean shear rate, so that nonlinear effects are important.

PACS. 47.27.Eq Turbulence simulation and modeling – 47.27.Gs Isotropic turbulence; homogeneous turbulence

1 Introduction

Compressible turbulence modeling is an essential element for calculations of many problems of practical engineering interest, such as combustion, environment and aerodynamics. The direct extension of incompressible models were used in the calculation of turbulent flows at moderate turbulent Mach number. This extension should be considerable success in the calculations, even if the turbulent Mach number was smaller. However, it failed to predict correctly the reduced growth rate of the turbulent kinetic energy when the compressibility was predominant [1–5]. This extension was observed in the modeling of two dilatational terms arise due to compressibility appearing in the turbulent kinetic energy transport equation.

Sarkar et al. [6] have proposed a model for the dilatational part of the total dissipation due to the divergence of velocity and another for the pressure dilatation [7] which represents a reversible transfer of energy between kinetic and internal energy. These algebraic models are obtained from an asymptotic analysis that is formally valid for small turbulent Mach number. Zeman [8] proposes that the dilatational part of the total dissipation becomes progressively important as the turbulent Mach number increases due to the appearance of eddy shocklets; he models dilatational dissipation as proportional to the solenoidal dissipation and a function of the turbulent Mach number. The studies of Speziale et al. [5] and Adumitroaie et al. [9] have demonstrated that the dilatational effects on homogeneous shear flow are, in fact, much smaller than one believes; consequently, they do not appear to be reflecting the correct physics of the reduced growth rate. It appears from DNS results [10,11], that the phenomenon responsible for the reduced growth rate is due to the reduction in the Reynolds shear stress anisotropy. This effect is thought to be due to the effects of compressibility on the pressure-strain correlation. This establishes the motivation of the present work. The main theme of this paper is to extend the incompressible Launder, Reece and Rodi model [12] for the pressure-strain correlation to compressible turbulent flow, in which the C_1, C_3 and C_4 coefficients become dependent on the turbulent Mach number. The validity of the proposed model has been tested for four selected cases from the DNS results of Sarkar [10] for compressible homogeneous shear flow.

The organization of this paper is as follows: In Section 2, the Favre-average forms of the governing equations are given. The development of closures for the effects of compressibility in the Reynolds stress equations is described in Section 3. An expression of the temporal growth rate of the kinetic energy can be used to construct a compressible correction to the L.R.R model [12]. The results are discussed in Section 4.

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2 Governing equations

In this section, we focus on the derivation of the evolution equations for the turbulent quantities. To this end, we first recall the Navier-Stokes equations for compressible fluids.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_j} = 0 \qquad (1)$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad (2)$$

where:

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_l}{\partial x_l}\delta_{ij}\right),$$

is the viscous stress tensor.

In these equations, the u_i are the components of the velocity, p, ρ, and μ are respectively the pressure, the density and the molecular viscosity. Any flow variable f can be decomposed into ensemble mean and fluctuating parts as follows:

$$f = \overline{f} + f^{'}, \qquad (3)$$

where, for homogeneous turbulence, the mean $\overline{f}$ can be taken to be a spatial average or, for a statistically steady turbulence, it can be taken to be a time average. An alternative decomposition based on a mass-weighted averages can be used wherein:

$$f = \widetilde{f} + f^{''}, \qquad (4)$$

given that $\widetilde{f}$ is the Favre-average which is defined as $\widetilde{f} = \frac{\overline{\rho f}}{\overline{\rho}}$. As in the traditional studies of compressible Reynolds stress, both (3) and (4) will be used. A direct averaging of equations (1–2) yields the mean continuity and momentum equations which are as follows:

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{u}_i)}{\partial x_j} = 0 \qquad (5)$$

$$\frac{\partial(\overline{\rho}\widetilde{u}_i)}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{u}_i\widetilde{u}_j)}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\sigma}_{ij}}{\partial x_i} - \frac{\partial(\overline{\rho}\widetilde{u_i^{''}u_j^{''}})}{\partial x_j}. \qquad (6)$$

In order to achieve closure, we need a model for the Favre-average Reynolds stress $\widetilde{u_i^{''}u_j^{''}}$. For homogeneous shear flow problem where the mean density and the gradient of the mean velocity are constant, the Favre-averaged Reynolds stress tensor is a solution of the transport equations:

$$\frac{d\widetilde{u_i^{''}u_j^{''}}}{dt} = -\widetilde{u_i^{''}v^{''}}S\delta_{j1} - \widetilde{u_j^{''}v^{''}}S\delta_{i1} + \frac{\phi_{ij}}{\overline{\rho}} + \frac{2}{3}\left(\frac{\overline{p^{'}d^{'}}}{\overline{\rho}} - \epsilon\right)\delta_{ij}, \qquad (7)$$

where S, ϕ_{ij}, ϵ and $\overline{p^{'}d^{'}}$ are respectively the mean shear rate, the deviatoric part of the pressure-gradient velocity correlation, the total dissipation rate tensor and the pressure-dilatation correlation.

The contraction of (7) yields the turbulent kinetic energy equation:

$$\frac{dk}{dt} = P - \epsilon + \frac{\overline{p^{'}d^{'}}}{\overline{\rho}}, \qquad (8)$$

where $k = \frac{1}{2}\widetilde{u_i^{''}u_i^{''}}$ is the Favre-averaged turbulent kinetic energy and $P = -\widetilde{u^{''}v^{''}}S$ is the turbulence production. Sarkar et al. [6] and Zeman [8] decompose the turbulent dissipation into solenoidal and dilatational parts as follows: $\epsilon = \epsilon_s + \epsilon_c$, where for homogeneous turbulence $\overline{\rho}\epsilon_s = \overline{\mu\omega_i^{'}\omega_i^{'}}$, $\overline{\rho}\epsilon_c = \frac{4}{3}\overline{\mu(u_{i,i}^{'})^2}$ are respectively, the solenoidal and compressible parts of the turbulent dissipation rate given that $\omega_i^{'}$ is the fluctuating vorticity. ϵ_s represents the turbulent dissipation arising from the traditional energy cascade which is solenoidal, ϵ_c represents the turbulent dissipation arising from compressible modes. The transport equation for the solenoidal dissipation is of the form [13]:

$$\frac{d\epsilon_s}{dt} = C_{\epsilon_1}\frac{\epsilon_s}{k}P - C_{\epsilon_2}\frac{{\epsilon_s}^2}{k}. \qquad (9)$$

In compressible homogeneous shear flow, this equation is identical to its incompressible counterpart. In (9) C_{ϵ_1}, C_{ϵ_2} are constants ($C_{\epsilon_1} = 1.44$, $C_{\epsilon_2} = 1.83$).

Hence, a full Reynolds stress closure is achieved in compressible turbulence if models are provided for:

(i) the pressure-dilatation correlation $\overline{p^{'}d^{'}}$
(ii) the compressible dissipation ϵ_c
(iii) the deviatoric part of the pressure-strain correlation.

For the pressure-dilatation and the compressible dissipation, we have retained models of Sarkar [6,7]. These models are algebraic and take in homogeneous shear flow the following simple forms:

$$\overline{p^{'}d^{'}} = -\overline{\rho}\alpha_1 P M_t + \alpha_2\overline{\rho}M_t^2\epsilon_s \qquad (10)$$

$$\epsilon_c = \alpha M_t^2\epsilon_s \qquad (11)$$

where α, α_1, α_2 are constants that take on the values of 0.5, 0.15 and 0.2 respectively. M_t denotes the turbulent Mach number that is $M_t = \frac{u}{c}$, is a solution of the transport equation [5]:

$$\frac{dM_t}{dt} = M_t\frac{P}{2k} + \frac{M_t}{2\overline{\rho}k}\left[1 + \frac{1}{2}\gamma(\gamma-1)M_t^2\right](\overline{p^{'}d^{'}} - \overline{\rho}\epsilon), \qquad (12)$$

where u is the r.m.s velocity fluctuation, c is the speed of sound and γ is the ratio of specific heats. Finally, in order to close these equations, a model for the pressure-strain correlation is needed. It is assumed that the proposed model for ϕ_{ij} will be functionally the same as the incompressible L.R.R model, except for new coefficients C_1, C_3 and C_4 that will become a function of the turbulent Mach number.

3 Compressible closure for the pressure-strain covariance

3.1 Literature models

The modeling of the pressure-strain correlation constitutes a more complicated problem of closure, this is due to the complexity of phenomena that they reflect and to the role that they play in the mechanism of redistribution of the energy between the different components of the Reynolds stress tensor.

The modeling of this tensor must take into account two different physical mechanisms appearing in the analysis of the different terms of the Poisson equation for the fluctuating pressure [13] which is obtained by taking the divergence of the Navier-Stokes equations. This equation contains two terms, the first one arises from the mean rate of strain and its interaction with the turbulence. The second term is generated from a mutual interaction between turbulence components. Models proposed for the pressure-strain correlation must reflect these two mechanisms, implied in the field of the fluctuating pressure. Thus, models will constitute a fast part (linear part) and a slow part which describes the return to the isotropy behaviour of turbulence. Rotta [14] has developed the first simple model for the nonlinear part. This model has served as a cornerstone for the representation of the slow pressure-strain in a variety of the commonly used second-order closure such as the Launder, Reece and Rodi model [12]. Subsequent to this work, Lumley [15] has demonstrated the need for nonlinear terms in models for the slow part of the pressure-strain correlation and derived a nonlinear representation theorem for this correlation based on isotropic tensor function theory. The simplest model for the rapid part is proposed by Rotta [16]. This model is based on the assumption of isotropy of the coefficients of the mean velocity gradients. Starting with the work of Launder et al. [12], anisotropy models for the rapid pressure-strain correlation have been formulated wherein the coefficients of the mean velocity gradients are taken to be functions of the anisotropy tensor. In the Launder et al. model, the fourth-rank tensor of coefficients is linear in the anisotropy tensor, whereas most of the newer models developed during the last decade are nonlinear such as models proposed by Shih and Lumley [17], Haworth and Pope [18], Speziale [19] and Launder and Tselepidakis [20].

In compressible turbulence, the modeling of this tensor articulates on the simple extension of models established in incompressible turbulence as mentioned in the works of Adumitroaie et al. [9] and Fujihiro Hamba [21]. To account for the non-vanishing fluctuating dilatation in compressible flows, Adumitroaie has developed a compressible correction depending on the magnitude of the turbulent Mach number to the L.R.R model. The effects of compressibility using the new representation are consistent with DNS results of compressible mixing layers. For Fujihiro Hamba, the compressibility effect can be reproduced in terms of the parameter of normalized pressure variance. The extension of Fujihiro Hamba for the incompressible pressure-strain model is directly related to the pressure variance.

Table 1. Initial conditions.

case	$M_{g,0}$	$M_{t,0}$	$(\frac{Sk}{\epsilon_s})_{,0}$	$b_{11,0}$	$b_{22,0}$	$b_{12,0}$
A_1	0.22	0.4	1.8	0	0	0
A_2	0.44	0.4	3.6	0	0	0
A_3	0.66	0.4	5.4	0	0	0
A_4	1.32	0.4	10.8	0	0	0

We have already said, that the present work articulates a simple extension of the L.R.R model established in incompressible situations. In spite of its simplicity, this model is unable to reproduce relatively correct incompressible turbulence behavior, as mentioned in the asymptotic equilibrium values of the Reynolds stress components, the DNS results of Rogers [22] for the incompressible homogeneous shear flow have showed that the LRR model underpredicts the magnitude of the normal Reynolds stress anisotropies b_{11}, b_{22} and b_{33} by about 26%, 21% and 45% respectively (Tab. 2). On the other hand, it overpredicts the asymptotic equilibrium value of the Reynolds shear stress anisotropy by about 15.7%. Although the standard model of L.R.R still needs to be improved to explain incompressible DNS results. The same model with variable density extension has been extended to be tested by several authors for compressible homogeneous shear flow. Speziale et al. [5,23] have shown that a variable density extension of the L.R.R model in conjunction with the compressible dissipation and pressure-dilatation models of Sarkar [6,7] cannot properly reproduce the DNS results of Sarkar et al. [24] of the turbulent kinetic energy for the initial conditions $\frac{Sk}{\epsilon_s} = 7.18$, $R_\lambda = 15$ and $M_t = 0.2$. and its unable to predict the dramatic changes in the Reynolds stress anisotropies that arise from compressibility effects. This can be seen more clearly in Table 3, where the model predictions for the equilibrium Reynolds stress anisotropies are compared with DNS for compressible homogeneous shear flow.

3.2 Extension of the L.R.R model

The study of compressibility effects on the turbulent homogeneous shear flow behavior made these last years the objective of several researches; as mentioned in the works of Blaisdell et al. [25,26], Sarkar et al. [6,7,10], Adumitroaie et al. [9] and Fujihiro Hamba [21]. The direct numerical results developed by Sarkar [10] and Fujihiro Hamba [21] show that the temporal growth rate of the turbulent kinetic energy ($\Lambda = \frac{1}{Sk}\frac{dk}{dt}$) is extensively influenced by compressibility. Therefore this term could be our starting point to extend the Launder, Reece and Rodi model for the pressure-strain correlation to compressible turbulent flows.

We recall that for homogeneous shear flow, the temporal growth rate of the turbulent kinetic energy is

Table 2. Comparison of the L.R.R model predictions for the equilibrium Reynolds anisotropies with the DNS results of Rogers et al. [22] for incompressible homogeneous shear flow.

Equilibrium values	Launder, Reece and Rodi model	DNS results of Rogers
b_{11}	0.155	0.215
b_{12}	–0.187	–0.158
b_{22}	–0.121	–0.153
b_{33}	–0.034	–0.062

Table 3. Comparison of the L.R.R model predictions (using the dilatational terms of Sarkar [6,7]) for the equilibrium Reynolds anisotropies with the DNS results of Blaisdell et al. [5] for the compressible homogeneous shear flow.

Equilibrium values	Launder, Reece and Rodi model	DNS results of Blaisdell
b_{11}	0.166	0.424
b_{12}	–0.187	–0.118
b_{22}	–0.130	–0.236
b_{33}	–0.036	–0.188

defined by:

$$\Lambda = \frac{1}{Sk}\frac{dk}{dt} = -2b_{12} - \frac{\epsilon_s}{Sk} + \frac{\frac{\overline{p'd'}}{\overline{\rho}} - \epsilon_c}{Sk}. \qquad (13)$$

For a mixing layer, were the turbulence is isotropic, with the assumption that we can neglect the pressure-dilatation term and the dissipation anisotropy in the kinetic energy equation, Vreman et al. [11] have pointed out that there is an approximate proportionality between the growth rate and the diagonal of the rapid pressure-strain terms:

$$\Lambda = \beta_1\frac{\phi_{11}^r}{Sk} = \beta_2\frac{\phi_{22}^r}{Sk}. \qquad (14)$$

Two compressible direct simulations in which the initial value of turbulent Mach number was set to M_t=0.1, M_t=0.3 and $\frac{Sk}{\epsilon_s}$=7.1 were performed by Fujihiro Hamba [21]. The analysis of these results show that Λ normalized by its incompressible value is proportional to $\frac{\phi_{11}}{Sk}$ and $\frac{\phi_{22}}{Sk}$ normalized respectively by their incompressible parts $(\frac{\phi_{11}}{Sk})^I$ and $(\frac{\phi_{22}}{Sk})^I$.

$$\frac{\Lambda}{\Lambda^I} \simeq \frac{\frac{\phi_{11}}{Sk}}{(\frac{\phi_{11}}{Sk})^I} \simeq \frac{\frac{\phi_{22}}{Sk}}{(\frac{\phi_{22}}{Sk})^I}. \qquad (15)$$

Sarkar [10] achieved two series of simulations in which M_t and M_g vary independently. From these series as one can notice, there is a systematic increase in the magnitude of the streamwise and transverse anisotropies when the turbulent Mach number and the gradient Mach number increase separately

$$\frac{\Lambda}{\Lambda^I} \simeq \frac{b_{11}^I}{b_{11}} \simeq \frac{b_{22}^I}{b_{22}}. \qquad (16)$$

It can be deduced that the pressure-strain correlation tensor in homogeneous shear flow is significantly changed due to compressibility. Let us recall now the components of the deviatoric part of the pressure-strain correlation (L.R.R).

$$\begin{aligned}\phi_{11} &= -\overline{\rho}C_1\epsilon_s b_{11} + \left(\frac{C_3}{3} + C_4\right)\overline{\rho}kSb_{12}\\ \phi_{22} &= -\overline{\rho}C_1\epsilon_s b_{22} + \left(\frac{C_3}{3} - C_4\right)\overline{\rho}kSb_{12}\\ \phi_{12} &= -\overline{\rho}C_1\epsilon_s b_{12} + \frac{C_2}{2}\overline{\rho}kS + \frac{C_3 - C_4}{2}\overline{\rho}kb_{11}S\\ &\quad + \frac{C_3 + C_4}{2}\overline{\rho}kb_{22}S. \end{aligned} \qquad (17)$$

Using equation (14) we shall write:

$$\begin{aligned}\frac{\Lambda}{\Lambda^I} &\simeq \frac{\frac{\phi_{11}^r}{Sk}}{(\frac{\phi_{11}^r}{Sk})^I} \simeq \frac{\frac{\phi_{22}^r}{Sk}}{(\frac{\phi_{22}^r}{Sk})^I}\\ \frac{\Lambda}{\Lambda^I} &\simeq \frac{\frac{\phi_{11}^r}{Sk} - \frac{\phi_{22}^r}{Sk}}{(\frac{\phi_{11}^r}{Sk})^I - (\frac{\phi_{22}^r}{Sk})^I} = \frac{\frac{\phi_{11}^r}{Sk} + \frac{\phi_{22}^r}{Sk}}{(\frac{\phi_{11}^r}{Sk})^I + (\frac{\phi_{22}^r}{Sk})^I}\\ \frac{\Lambda}{\Lambda^I} &\simeq \frac{C_3 b_{12}}{C_3^I b_{12}^I} = \frac{C_4 b_{12}}{C_4^I b_{12}^I}.\end{aligned}$$

For homogeneous flow, the Helmoltz decomposition gives a unique split of the velocity into incompressible and compressible components, that is $u = u^I + u^c$, where $u_{i,i}^I = 0$ and $\epsilon_{ijk}u_{k,j}^c = 0$. This decomposition permits to write b_{12} as follows:

$$b_{12} = (1 - \chi_k)b_{12}^I + \chi_k b_{12}^c,$$

where $b_{12}^I = \frac{\widetilde{u''^I v''^I}}{2k^I}$, $b_{12}^c = \frac{(\widetilde{u''v''} - \widetilde{u''^I v''^I})}{2k^c}$, $\chi_k = \frac{k^c}{k}$, $k^I = \frac{\widetilde{u_i''^I u_i''^I}}{2}$ and $k^c = \frac{\widetilde{u_i''^c u_i''^c}}{2}$.

According to DNS results of Sarkar [10], the compressibility coefficient χ_k is weak and the Reynolds shear stress anisotropy can be rewritten as: $b_{12} \simeq b_{12}^I(1 - \chi_k)$.

To obtain the compressibility parameter model of χ_k, we assume that the time scale of solenoidal field $\frac{k^I}{\epsilon_s}$ is nearly equal to that of the dilatational field: $\frac{k^I}{\epsilon_s} = \frac{k^c}{\epsilon_c}$. This relation according to Sarkar model [6] for compressible dissipation gives: $\chi_k = \frac{\alpha M_t^2}{1+\alpha M_t^2}$. These approximations permit to obtain for C_3 and C_4 the following expressions:

$$C_3 = C_3^I \frac{\Lambda}{\Lambda^I}\left(1+\alpha M_t^2\right) \text{ and } C_4 = C_4^I \frac{\Lambda}{\Lambda^I}\left(1+\alpha M_t^2\right)$$

Using equations (14), (15) and (16), we can write:

$$\frac{\Lambda}{\Lambda^I} \simeq \frac{\frac{\phi_{11}}{Sk}}{(\frac{\phi_{11}}{Sk})^I} \simeq \frac{\frac{\phi_{11}^r}{Sk}}{(\frac{\phi_{11}^r}{Sk})^I}$$

$$\frac{\Lambda}{\Lambda^I} \simeq \frac{\frac{\phi_{11}}{Sk} - \frac{\phi_{11}^r}{Sk}}{(\frac{\phi_{11}}{Sk})^I - (\frac{\phi_{11}^r}{Sk})^I} = \frac{\frac{\phi_{11}^s}{Sk}}{(\frac{\phi_{11}^s}{Sk})^I} = \frac{C_1}{C_1^I}\frac{\Lambda^I}{\Lambda}\frac{\frac{\epsilon_s}{P}b_{12}}{(\frac{\epsilon_s}{P})^I b_{12}^I}. \quad (18)$$

The DNS of Sarkar [10] show that the relative dissipation $\frac{\epsilon_s}{P}$ is less affected by compressibility. This permit to approximate $\frac{\epsilon_s}{P}/\left(\frac{\epsilon_s}{P}\right)^I$ by 1. Then we may write from (18):

$$C_1 = C_1^I \left(\frac{\Lambda}{\Lambda^I}\right)^2 (1+\alpha M_t^2)$$

$\frac{\Lambda}{\Lambda^I}$ can be expressed as follows:

$$\frac{\Lambda}{\Lambda^I} = \frac{2b_{12}[\alpha_1 M_t - 1]}{[-2b_{12} - \frac{\epsilon_s}{Sk}]^I} + \frac{\frac{\epsilon_s}{Sk}\left[(\alpha_2-\alpha)M_t^2 - 1\right]}{\left[-2b_{12} - \frac{\epsilon_s}{Sk}\right]^I},$$

this relation can be recast in a more compact form when we use the Helmoltz decomposition:

$$\frac{\Lambda}{\Lambda^I} = \left[\frac{\alpha_1 M_t - 1}{(\frac{\epsilon_s}{P})^I - 1} + \frac{(\alpha_2-\alpha)M_t^2 - 1}{(\frac{P}{\epsilon_s})^I - 1}\right](1-\chi_k).$$

Assuming $\left(\frac{\epsilon_s}{P}\right)^I$ by its incompressible equilibrium value $(C_{\epsilon_1}-1)/(C_{\epsilon_2}-1)$, we have:

$$\frac{\Lambda}{\Lambda^I} = \frac{1 - aM_t - bM_t^2}{1+\alpha M_t^2},$$

the calibration of a and b based on direct numerical simulations of Sarkar [10] for homogeneous shear flow gives for C_1, C_3 and C_4 the following expressions:

$$C_1 = \frac{C_1^I}{1+\alpha M_t^2}(1-0.44M_t)^2$$
$$C_3 = C_3^I(1-1.5M_t^2)$$
$$C_4 = C_4^I(1-0.5M_t).$$

Application of these corrections allows the pressure-strain model to be written as:

$$\phi_{ij} = -C_1^I\frac{(1-0.44M_t)^2}{(1+\alpha M_t^2)}\overline{\rho}\epsilon_s b_{ij} + C_2^I\overline{\rho}k\widetilde{S}_{ij}$$
$$C_3^I(1-1.5M_t^2)\overline{\rho}k\left[b_{ik}\widetilde{S}_{jk} + b_{jk}\widetilde{S}_{ik} - \frac{2}{3}b_{mn}\widetilde{S}_{mn}\delta_{ij}\right]$$
$$+C_4^I(1-0.5M_t)\overline{\rho}k[b_{ik}\widetilde{\Omega}_{jk} + b_{jk}\widetilde{\Omega}_{ik}] \quad (19)$$

where $\widetilde{S}_{ij} = \frac{1}{2}(\frac{\partial\tilde{u}_i}{\partial x_j} + \frac{\partial\tilde{u}_j}{\partial x_i})$, $\widetilde{\Omega}_{ij} = \frac{1}{2}(\frac{\partial\tilde{u}_i}{\partial x_j} - \frac{\partial\tilde{u}_j}{\partial x_i})$, $b_{ij} = \frac{\widetilde{u_i'' u_j''} - \frac{2}{3}k\delta_{ij}}{2k}$ are, respectively, the mean rate of strain tensor, the mean vorticity tensor and the anisotropy tensor. The coefficients C_i^I come from the incompressible pressure-strain model.

4 Numerical results

4.1 Initial conditions

The nonlinear ordinary differential equations (7), (8), (9) and (12) are solved subjects to the initial conditions (Tab. 1).

The DNS results of Sarkar [10] emphasize the importance of a new parameter of compressibility noted by M_g, that can be an important parameter to understand the compressibility effect on the turbulence. We recall that the three time scales in homogeneous compressible turbulence : the mean distortion time scale,

$$\tau_d^{-1} = (\widetilde{u}_{i,j}\widetilde{u}_{i,j})^{\frac{1}{2}}$$

(where $\widetilde{u}_{i,j}$ is the mean velocity gradient), the turn-over time,

$$\tau_t^{-1} = \frac{\sqrt{2k}}{l}$$

(where k is the turbulent kinetic energy and l is the length scale of the energy containing eddies) and the acoustic time scale,

$$\tau_c^{-1} = \frac{c}{l}$$

(where c is the speed of sound), permit to construct the turbulent Mach number and the gradient Mach number. The turbulent Mach number is the ratio of the acoustic time scale to the turn-over time: $M_t = \frac{\tau_a}{\tau_t}$. Sarkar [10], Durbin and Zeman [27], Cambon et al. [28] and Jacquin et al. [29] have defined the gradient Mach number M_g as being the ratio of the acoustic time scale to the mean distortion time scale: $M_g = \frac{\tau_a}{\tau_d}$. The ratio of the gradient Mach number to the turbulent Mach number noted by r defines the rapidity of the distortion, for r $\gg 1$ (turn-over time is much larger than the distortion time scale), the nonlinear interactions within the turbulence can be neglected (rapid distortion theory: RDT). Table 1 represents the initial conditions of the DNS of Sarkar [10] for homogeneous shear flow, which are considered by their author outside the application of the rapid distortion theory, because the choice of the integral length scale gives weak values of M_g and r. The last two simulations A_3 and A_4 are examined by Simone et al. [30]. they defined the gradient Mach number using the length scale $l = \frac{(2k)^{\frac{3}{2}}}{\epsilon_s}$, which conducts to large values of (M_g, r) ((6.6, 16.5) for A_3 and (13.2, 33) for A_4). From these values they emphasized that A_3 and A_4 correspond to the limit of RDT.

4.2 Discussion

The transport equations (7), (8), (9) and (12)-incorporating the model of the pressure-strain correlation discussed in section (3)-are solved numerically for compressible homogeneous shear flow using a fourth-order

Table 4. Comparison of the present model predictions for the long-time values of the anisotropy tensor for cases A_1, A_2 and A_3 with the DNS of Sarkar [10] and the formulas of Stefan [31]

Long-time values	Present Model	DNS of Sarkar	Formulas of Stefan
b_{11}	0.342, 0.410, 0.525	0.32, 0.44, 0.51	0.370, 0.447, 0.504
b_{12}	–0.138, –0.125, –0.1	–0.145, –0.12, –0.092	–0.139, –0.114, –0.093
b_{22}	–0.211, –0.236, –0.273	–0.2, –0.24, –0.275	–0.207, –0.240, –0.264

Runge-Kutta numerical integration scheme. Four simulations labelled respectively A_1, A_2, A_3 and A_4, are performed. In these simulations the gradient Mach number $M_{g,0}$ increases respectively in cases A_1 to A_4 by changing the initial value of $\frac{Sk}{\epsilon_s}$, taking the initial value of M_t constant (Tab. 1). The DNS data of Sarkar [10] are presented with circles, squares, triangles and diamonds in order of increasing $M_{g,0}$. First, we will consider model predictions for the case where there are no corrections of C_1, C_3 and C_4 in order to assess the performance of the proposed correction. From the results plotted in Figures 1, 2, 3 and 4, it is clear that the standard model in conjunction with explicit dilatational terms proposed by Sarkar [6,7] is unable to predict the dramatic changes in the Reynolds stress anisotropies and the pressure-strain correlation that arise from compressibility. It is also found that the dilatational terms appear much smaller to reflect the phenomena of compressibility. The results in figure (1) as the normalized production term $-2b_{12}$ versus a dimensionless time $t^* = St$; it is clear that there is a decrease in the magnitude of the normalized production term $-2b_{12}$ when $M_{g,0}$ increases. The effect of compressibility on the other components is also of interest. Figures 2a-b show that there is an increase in the transverse and streamwise anisotropies from case A_1 to A_3. As one can remark that for $St = 20$, the turbulence eventually evolves to approximately constant values of the components of the anisotropy tensor. Contrary to the L.R.R's model, predictions of asymptotic values by our model, has the same tendency as the DNS results. This can seen more clearly from Table 4, where the model predictions for the long-time values of the anisotropy tensor b_{ij} are compared with those given by the formulas established by Stefan [31]. By these formulas, the components b_{11}, b_{12} and b_{22} are functions of the gradient Mach number:

$$b_{11} = \frac{2}{3} - 0.4\exp(-0.3M_g)$$
$$b_{12} = -0.17\exp(-0.2M_g)$$
$$b_{22} = -\frac{1}{3} + 0.17\exp(-0.3M_g).$$

Figures 3a, b, c show the historical time of the components of the pressure-strain ϕ_{11}, ϕ_{22}, and ϕ_{12}, overall agreement between the present model and DNS is good. In Figures 4, the present model predictions for the time evolution of $\frac{\epsilon_s}{Sk}$ and the relative dissipation $\frac{\epsilon_s}{P}$ are displayed. Figure 4b shows that there is an increase in the relative dissipation when $M_{g,0}$ increases. It is clearly seen that the primary reason of the decrease in $\frac{\epsilon_s}{Sk}$ ($\frac{\epsilon_s}{Sk} = -2b_{12}\,\frac{\epsilon_s}{P}$) is the reduced level of the production term (Fig. 1, Fig. 4b).

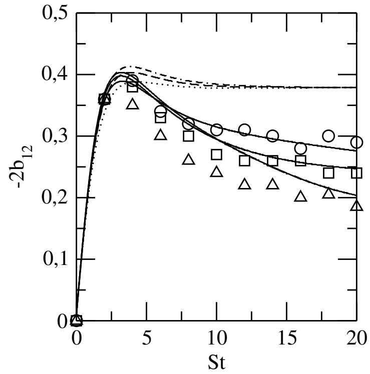

Fig. 1. Evolution of normalized production term $-2b_{12}$ in cases A_1, A_2 and A_3 for compressible homogeneous shear flow: Standard model(dots, dashed line, dot-dashed line); present model (line); DNS results of Sarkar [10] (circle, square, triangle).

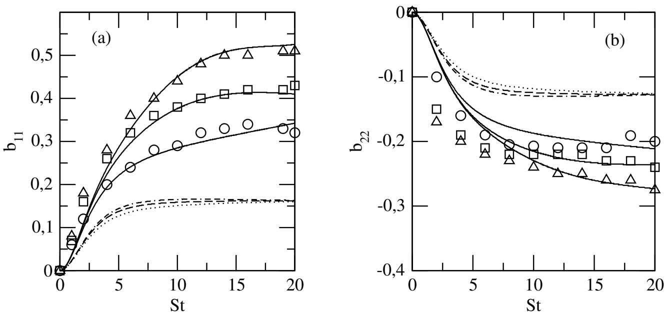

Fig. 2. Evolution of (a) the streamwise and (b) the transverse Reynolds stress anisotropy in cases A_1, A_2 and A_3 for compressible homogeneous shear flow.

Figure 5 shows the behaviour of the dilatational terms. It will be shown that these terms are much smaller to explain the compressibility effect on the turbulence. Using equation (13), one can notice that the compressibility effect of decreased growth rate of turbulent kinetic energy is due to a decrease of the normalized production term. It will be shown from cases A_1, A_2 and A_3 that the asymptotic values of turbulent parameters are highly dependent on the initial conditions when $M_{g,0}$ is changed. This shows that M_g is an important parameter that describes the level of stabilizing effect of compressibility. In Figures 6a, b, c, we present the components of the pressure-strain correlation for case A_4. These results show that the present model predictions are in disagreement with DNS results of Sarkar. This disagreement can be explained by:

(i) the standard hierarchy of the pressure-strain models remind us of a strong approximation for compressible

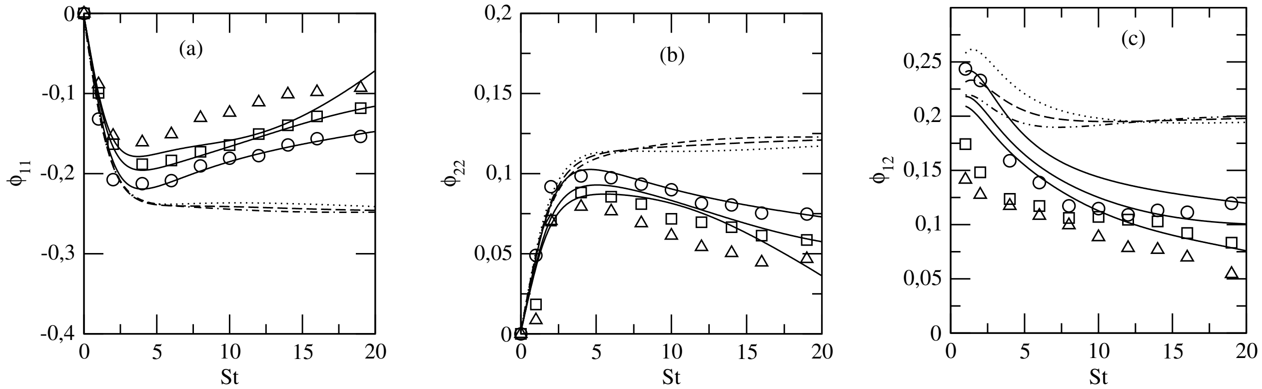

Fig. 3. Evolution of the pressure-strain correlation (a) ϕ_{11}, (b) ϕ_{22} and (c) ϕ_{12} in cases A_1, A_2 and A_3 for homogeneous shear flow.

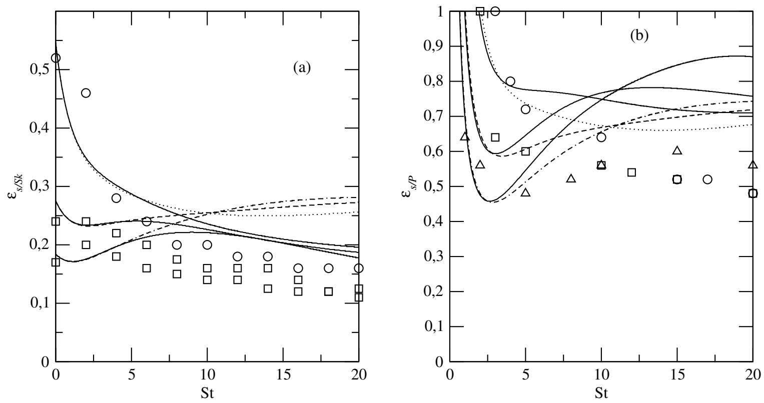

Fig. 4. Evolution of (a) $\frac{\epsilon_s}{Sk}$ and (b) the relative dissipation $\frac{\epsilon_s}{P}$ in cases A_1, A_2 and A_3 for homogeneous shear flow.

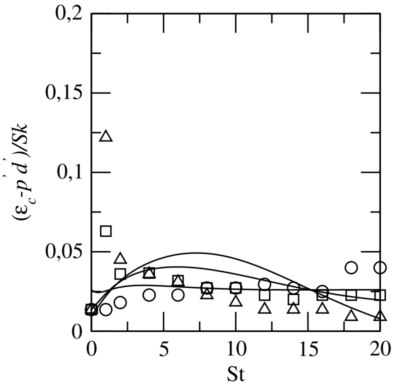

Fig. 5. Evolution of the dilatational terms in cases A_1, A_2 and A_3 for homogeneous shear flow.

flows, where the turbulent Mach number $M_t < 0.3$ that does not apply to high-speed compressible flows. While the DNS of Sarkar for case A_4 show that the turbulent Mach number achieves an equilibrium value of approximately 0.64.

(ii) using the integral length scale $l = \frac{(2k)^{\frac{3}{2}}}{\epsilon_s}$, A_4 corresponds to the limit of RDT, for this case the rapid redistribution terms are dominant. In the proposed formulation, C_1 is corrected while C_2 takes its incompressible value, thus leads to an extra contribution to the nonlinear part of the pressure strain-correlation. Therefore, it is interesting to have an open mind in the pursuit of our proposed approach modeling that appears to be very promising in predicting the unsteadiness of turbulent structures in A_4 of DNS [10].

5 Conclusions

The standard model for the pressure-strain correlation of L.R.R in conjunction with dilatational terms proposed by Sarkar yields poor predictions for compressible homogeneous shear flow. It was found that the dilatational terms are much smaller to reflect the correct physics of compressibility. An extension of the Launder, Reece and Rodi model has been proposed, the coefficients C_1, C_3 and C_4 can be functions of the turbulent Mach number. Application of the present formulation on compressible homogeneous shear flow has shown a broadly satisfactory agreement with DNS data of Sarkar for cases A_1, A_2 and A_3. The numerical simulations show that compressibility effects are highly dependent on the initial conditions when $M_{g,0}$ is varied; it shows that M_g can be a very important parameter to understand the phenomena of compressibility. Disagreement between calculations and DNS results of Sarkar observed in case A_4 will be explained by: i) failure of standard models of the pressure-strain term in

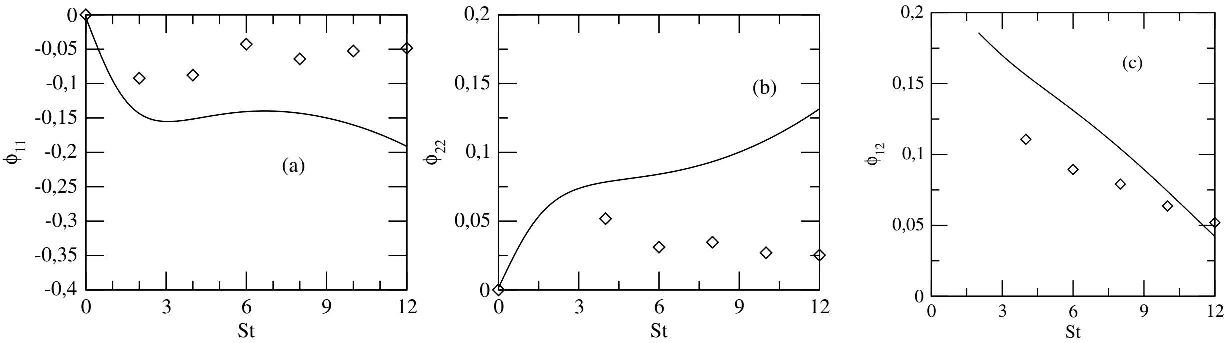

Fig. 6. Evolution of the pressure-strain correlation (a) ϕ_{11}, (b) ϕ_{22} and (c) ϕ_{12} in case A_4 for homogeneous shear flow.

high-speed compressible flows. For these flows, the pressure-strain term must be represented by models taking explicitly in account the compressibility effects in compressible turbulence situation. ii) models proposed by Sarkar for the dilatational terms are valid for small turbulent Mach number, that does not apply for these flows (case A_4). iii) Simone et al. have mentioned that A_3 and A_4 correspond to the limit of RDT, consequently the nonlinear interactions within the turbulence can be neglected. The correction adopted for C_1 can generate an elevated nonlinear part levels which is in contradiction with the rapid distortion theory.

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